ECON 3346: Portfolio Assignment

Group 8
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1 Introduction

The purpose of this project is to study the behavior of financial asset returns and to apply portfolio theory to construct efficient investment portfolios. Using historical market data, we analyze the statistical properties of asset returns, evaluate diversification benefits, and examine portfolio performance through risk-adjusted measures. The empirical analysis relies on publicly available financial data and is implemented using statistical software to ensure accurate and reproducible results.

1.1 Data Sources and Asset Selection

The dataset includes several financial assets representing different segments of the financial market in order to create a diversified investment universe. Specifically, the analysis considers the following securities:

- **Microsoft Corporation (MSFT)** – a large-cap technology company representing the equity sector.
- **iShares Core U.S. Aggregate Bond ETF (AGG)** – an exchange-traded fund providing broad exposure to the U.S. investment-grade bond market.
- **Vanguard Real Estate ETF (VNQ)** – an ETF that tracks real estate investment trusts (REITs), capturing the performance of the real estate sector.
- **SPDR S&P 500 ETF (SPY)** – used as a proxy for the overall stock market.
- **3-Month Treasury Bill Rate** – used as the risk-free benchmark in the portfolio and CAPM analysis.

Historical price data for the traded assets were obtained from Yahoo Finance, while the Treasury bill yield series was collected from the Bank of Canada database. The sample period spans from September 1, 2022 to August 31, 2025. Both daily and weekly price observations are used to construct return series for the empirical analysis.

1.2 Software and Analytical Approach

All computations are conducted using the `R` programming environment. The dataset is processed to compute continuously compounded returns, estimate summary statistics, and construct optimal portfolios using mean–variance methods. In addition, the analysis applies the Capital Asset Pricing Model (CAPM) to estimate asset betas and evaluate the relationship between expected return and systematic risk.

Figure 1 provides a visual summary of the relative performance of the selected assets over the full sample, illustrating the heterogeneity in return outcomes across equities, bonds, and real estate exposure.

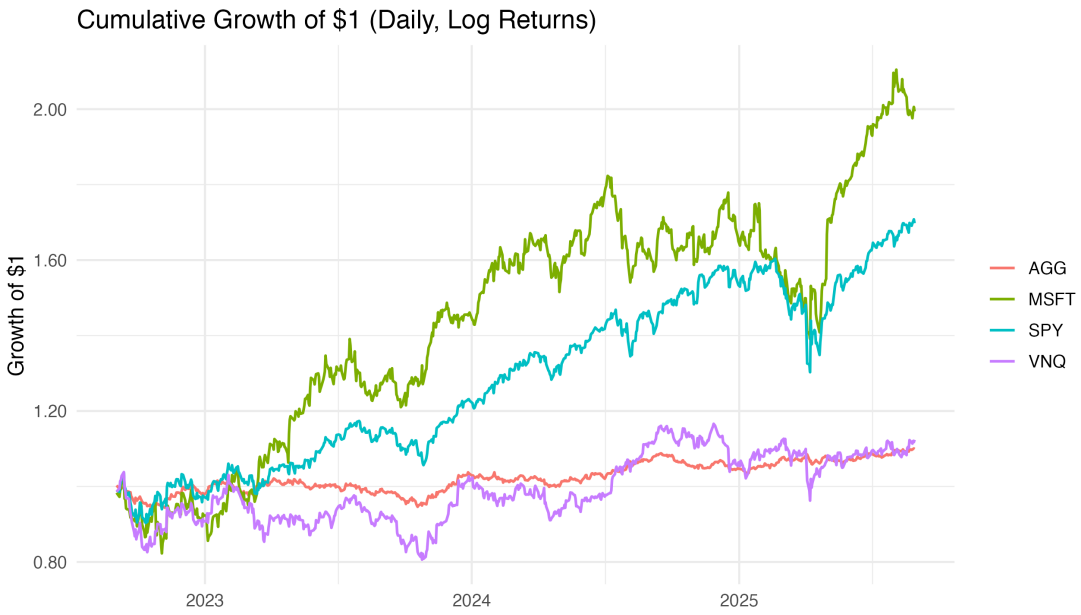

Figure 1: Cumulative growth of $1 invested in each asset using daily continuously compounded (log) returns over the sample period.

Part A

A1. Daily Prices to an Annual Returns Distribution

(a) Daily continuously compounded returns. For each asset, we compute the continuously compounded (log) daily return from adjusted closing prices:

$$r_t = \ln(P_{t+1}) - \ln(P_t).$$

(b) Daily expected return and standard deviation. Using the daily log-return series over 2024–09–01 to 2025–08–31(realized trading days: 2024–09–03 to 2025–08–29), we estimate the expected daily return and daily volatility using the sample mean and sample standard deviation of $\{r_t\}$.

(c) Annualization. Using 252 business days per year, annualized moments are computed as:

$$\mu_{\text{ann}} = 252\,\mu_{\text{daily}}, \qquad \sigma_{\text{ann}} = \sqrt{252}\,\sigma_{\text{daily}}.$$

Under a normal approximation for daily log returns, the annual log return (sum of daily log returns) is

$$R_{\text{ann}} = \sum_{t=1}^{252} r_t \approx \mathcal{N}\big(252\,\mu_{\text{daily}},\ 252\,\sigma^2_{\text{daily}}\big).$$

Table 1: Daily and Annualized Return Moments (Log Returns), 2024–09–03 to 2025–08–29

Asset	Mean (Daily)	SD (Daily)	Mean (Annual)	SD (Annual)
MSFT	0.0008110719	0.0155815880	0.2043901	0.2473501
AGG	0.0001227452	0.0031853230	0.0309318	0.0505654

A2. Are returns normally distributed?

To assess normality, we examine kurtosis and apply the Jarque–Bera (JB) test (null hypothesis: normality) using daily log returns over 2024–09–03 to 2025–08–29. The kurtosis of MSFT is 11.5399, indicating extremely fat tails relative to the normal benchmark of 3. The JB test strongly rejects normality for MSFT ($X^2 = 795.43$, $p < 2.2 \times 10^{-16}$). For AGG, kurtosis is 3.7735 (mild excess kurtosis), and the JB test also rejects normality at the 5% level ($X^2 = 8.1716$, $p = 0.01681$). Overall, normality is not a reasonable assumption for these daily returns, especially for MSFT.

The visual evidence in Figures 2–3 aligns with the test results: the return distributions exhibit noticeable tail behavior, and MSFT in particular shows strong departures from the normal benchmark in the extremes.

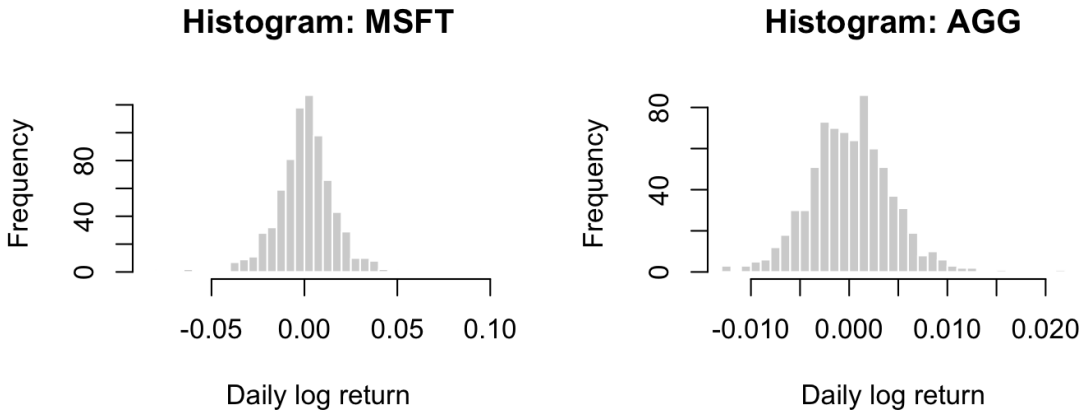

Figure 2: Histograms of daily log returns for MSFT and AGG over 2024–09–03 to 2025–08–29.

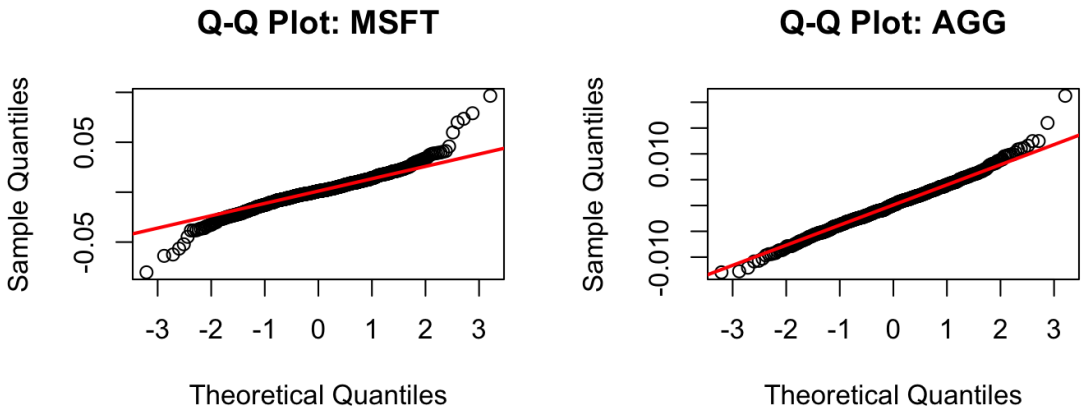

Figure 3: Normal Q–Q plots of daily log returns for MSFT and AGG over 2024–09–03 to 2025–08–29. Systematic deviations from the reference line indicate non-normality, particularly in the tails.

A3. Variance–Covariance Matrix (Daily Returns)

Let $r_t = (r_t^{MSFT}, r_t^{AGG})^\top$ denote the vector of daily continuously compounded (log) returns. The variance–covariance matrix is

$$\Sigma = \mathrm{Var}(r_t) = \begin{pmatrix} \mathrm{Var}(r^{MSFT}) & \mathrm{Cov}(r^{MSFT}, r^{AGG}) \\ \mathrm{Cov}(r^{AGG}, r^{MSFT}) & \mathrm{Var}(r^{AGG}) \end{pmatrix}.$$

Using the daily return series from 2024–09–03 to 2025–08–29, the estimated daily variance–covariance matrix is

$$\widehat{\Sigma} = \begin{pmatrix} 2.427859 \times 10^{-4} & 5.680540 \times 10^{-8} \\ 5.680540 \times 10^{-8} & 1.014628 \times 10^{-5} \end{pmatrix}.$$

For reference, the estimated daily correlation between MSFT and AGG is 0.001144523.

Figure 4 summarizes correlations across the full risky-asset set used later in A5–A8, highlighting the diversification role of fixed income (AGG) relative to equity-linked assets.

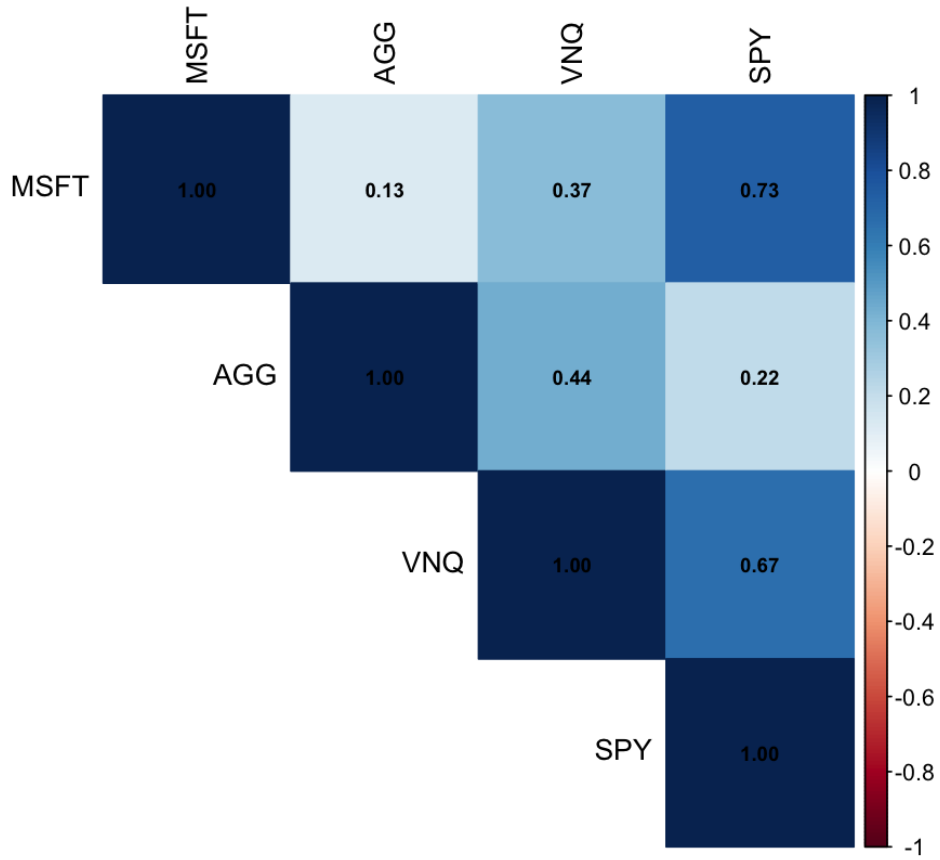

Figure 4: Correlation matrix of daily log returns for MSFT, AGG, VNQ, and SPY. Lower correlations (particularly involving AGG) indicate diversification benefits in mean–variance portfolio construction.

A4. Optimal Risky Portfolio P (Two Risky Assets)

Let $r_t = (r_t^{MSFT}, r_t^{AGG})^\top$ denote daily log returns, with sample mean vector $\hat{\mu}_d$ and sample covariance matrix $\widehat{\Sigma}_d$ computed from 2024–09–03 to 2025–08–29. Let the daily log risk-free rate be $r_{f,t}$, constructed from the Bank of Canada 3-month T-bill yield y_t (annual percent) via

$$r_{f,t} = \ln\left(1 + \frac{y_t/100}{252}\right),$$

and let $\bar{r}_{f,d}$ be the sample average of $r_{f,t}$ over the same window.

The tangency (maximum Sharpe) portfolio solves:

$$\max_{w_{\mathrm{MSFT}},\, w_{\mathrm{AGG}}} \frac{w^\top \hat{\mu}_d - \bar{r}_{f,d}}{\sqrt{w^\top \widehat{\Sigma}_d\, w}} \quad \text{s.t.} \quad w_{\mathrm{MSFT}} + w_{\mathrm{AGG}} = 1, \;\; w_{\mathrm{MSFT}}, w_{\mathrm{AGG}} \in [-0.5,\, 1.5],$$

where $w = (w_{\mathrm{MSFT}}, w_{\mathrm{AGG}})^\top$.

Table 2: A4: Optimal Risky Portfolio P (Max Sharpe), constrained weights in $[-0.5, 1.5]$ (2024–09–03 to 2025–08–29)

	w_{MSFT}	w_{AGG}	μ_d	σ_d	μ_{ann}	σ_{ann}
Optimal P	0.5109548	0.4890452	0.0004307939	0.0081629810	0.1085601	0.1295833

Over this window, the annualized risk-free rate is $r_{f,ann} \approx 0.03009157$ and the portfolio's annualized Sharpe ratio is approximately 0.6055448.

A5. Optimal Risky Portfolio with All Risky Assets and the Market Index

We now consider the four risky assets/market proxy (MSFT, AGG, VNQ, SPY) and the risk-free rate. Using daily continuously compounded returns over 2024–09–03 to 2025–08–29, we compute the sample mean vector $\hat{\mu}$ and covariance matrix $\hat{\Sigma}$ and form the unconstrained tangency (maximum Sharpe) portfolio:

$$w^* \propto \hat{\Sigma}^{-1}(\hat{\mu} - \bar{r}_{f,d}\mathbf{1}), \qquad \mathbf{1}^\top w^* = 1,$$

where $\bar{r}_{f,d}$ is the average daily log risk-free rate over the same window. (Unconstrained tangency portfolios can involve leverage and short positions.)

Table 3: A5: Unconstrained tangency (max Sharpe) portfolio weights (2024–09–03 to 2025–08–29)

Asset	MSFT	AGG	VNQ	SPY
Weight	0.1103730	0.85556621	-0.4406165	0.4745814

Table 4: A5: Portfolio moments and Sharpe ratio (annualization uses 252 trading days)

μ_d	σ_d	μ_{ann}	σ_{ann}	$r_{f,ann}$	Sharpe_{ann}
0.0004979695	0.0061790680	0.1254883	0.09808966	0.03009157	0.9725463

A6. Do the weights you computed in A5 make sense?

From A5, the unconstrained tangency (max Sharpe) portfolio weights over 2024–09–03 to 2025–08–29are:

$$w_{\text{MSFT}} = 0.1104, \quad w_{\text{AGG}} = 0.85557, \quad w_{\text{VNQ}} = -0.4406, \quad w_{\text{SPY}} = 0.4746, \qquad \sum_i w_i = 1.$$

At first glance, weights such as $w_{\text{AGG}} > 1$ and $w_{\text{VNQ}} < 0$ may look unusual, but they are reasonable for an *unconstrained* tangency portfolio (short-selling and leverage are allowed).

- **Why AGG is large and positive:** AGG has by far the lowest volatility in the sample (annualized $\sigma_{\text{AGG}} \approx 0.0506$), and it is only weakly correlated with the market proxy SPY. Even though its annualized mean return is modest ($\mu_{\text{AGG}} \approx 0.0309$), the combination of low risk and diversification benefits makes AGG an attractive component in a Sharpe-maximizing portfolio, which can lead to a weight exceeding 1 when leverage is permitted.
- **Why VNQ is negative:** VNQ has the lowest annualized mean in the sample ($\mu_{\text{VNQ}} \approx 0.00725$) while still having substantial volatility. In addition, VNQ is fairly correlated with SPY, so it behaves like an equity-like risk factor in this window. With a weak mean and meaningful comovement with the market, the optimizer can improve the overall Sharpe ratio by taking a short position in VNQ and reallocating risk toward combinations with better estimated risk-adjusted performance.
- **Why SPY and MSFT remain positive:** SPY has a strong annualized mean in the sample with moderate volatility. MSFT also has a high annualized mean but higher volatility and a strong correlation with SPY. Hence, the optimizer holds both, but does not place an extremely large weight on MSFT because its risk overlaps substantially with the broad market proxy.

Overall, the A5 weights make economic sense once we account for (i) unconstrained optimization allowing leverage and short positions, and (ii) the estimated sample moments: AGG provides low-risk diversification, VNQ exhibits weak sample performance with equity-like comovement, and SPY/MSFT capture the equity risk premium in this window.

A7. Optimal risky portfolio using a longer daily window (Sep 1, 2023 to Aug 31, 2025)

We repeat the A5 analysis, but now using daily continuously compounded (log) returns from September 1, 2023 to August 31, 2025. Using the four risky assets (MSFT, AGG, VNQ, SPY) and the daily log risk-free rate, we compute the sample mean vector $\hat{\mu}$ and covariance matrix $\hat{\Sigma}$ over the extended window and form the unconstrained tangency (maximum Sharpe) portfolio:

$$w^* \propto \hat{\Sigma}^{-1}(\hat{\mu} - \bar{r}_{f,d}\mathbf{1}), \qquad \mathbf{1}^\top w^* = 1,$$

where $\bar{r}_{f,d}$ is the sample average daily log risk-free rate over the same window.

Table 5: A7: Unconstrained tangency (max Sharpe) portfolio weights, daily data (Sep 1, 2023 to Aug 31, 2025).

Asset	MSFT	AGG	VNQ	SPY
Weight	0.0465687	0.5000654	-0.1913246	0.6446905

Table 6: A7: Portfolio moments and Sharpe ratio (annualization uses 252 trading days).

μ_d	σ_d	μ_{ann}	σ_{ann}	$r_{f,ann}$	Sharpe_{ann}
0.0005540859	0.0064225830	0.1396296	0.1019554	0.0393187	0.9838716

A8. Optimal risky portfolio using weekly data (Sep 1, 2023 to Aug 31, 2025)

We repeat the A5 analysis using *weekly* continuously compounded (log) returns from September 1, 2023 to August 31, 2025. Using the four assets (MSFT, AGG, VNQ, SPY) and the weekly log risk-free rate, we compute the sample mean vector $\hat{\mu}$ and covariance matrix $\hat{\Sigma}$ over this weekly window and form the unconstrained tangency (maximum Sharpe) portfolio:

$$w^* \propto \hat{\Sigma}^{-1}(\hat{\mu} - \bar{r}_f\mathbf{1}), \qquad \mathbf{1}^\top w^* = 1,$$

where $\bar{r}_f$ is the sample average *weekly* log risk-free rate over the same window.

Table 7: A8: Unconstrained tangency (max Sharpe) portfolio weights, weekly data (Sep 1, 2023 to Aug 31, 2025).

Asset	MSFT	AGG	VNQ	SPY
Weight	0.0535293	0.6199733	-0.3419542	0.6684515

Table 8: A8: Portfolio moments and Sharpe ratio (annualization uses 52 weeks per year).

μ_w	σ_w	μ_{ann}	σ_{ann}	$r_{f,ann}$	Sharpe_{ann}
0.002777909	0.01274279	0.1444513	0.09188953	0.04303887	1.103634

Consistency check (A5 vs A7 vs A8). A5 uses daily data for the most recent year (2024–09–03 to 2025–08–29), A7 uses daily data over a longer two-year window, and A8 uses weekly data over the same two-year window as A7. Across A7 and A8, the qualitative pattern is consistent: the portfolio places a large positive weight on AGG and SPY, a small positive weight on MSFT, and a negative weight on VNQ. These differences are expected because mean–variance portfolios are sensitive to estimated means and covariances; changing the sampling frequency (daily vs weekly) and the effective sample size can change the estimated inputs and therefore the resulting optimal weights.

A9. Robustness of the "optimum portfolio P of risky assets"

Mean–variance (tangency) portfolios are sensitive to estimated inputs (sample means and covariances). Therefore, a natural robustness check is to see how the optimal weights and Sharpe ratios change when we change (i) the sample window length and (ii) the data frequency.

Summary of our estimates.

- **A5 (daily, recent year):** $w = (0.1104,\ 1.0557,\ -0.6406,\ 0.4746)$ for (MSFT, AGG, VNQ, SPY), $\text{Sharpe}_{ann} \approx 0.9725$.
- **A7 (daily, two-year window):** $w = (0.0466,\ 0.5001,\ -0.1913,\ 0.6447)$, $\text{Sharpe}_{ann} \approx 0.9839$.
- **A8 (weekly, same two-year window):** $w = (0.0535,\ 0.6200,\ -0.3420,\ 0.6685)$, $\text{Sharpe}_{ann} \approx 1.1036$.

Overall conclusion. The optimal risky portfolio is *qualitatively robust* (similar long/short directions over the longer window), but *quantitatively not fully robust* (weights change with window length and sampling frequency).

A10. Risk-neutral investor: weight on the risky portfolio vs. the risk-free T-bill

Let P denote the optimal risky (tangency) portfolio and let r_f be the risk-free T-bill rate. Any complete portfolio that combines P with the T-bill can be written as

$$R_C = yR_P + (1-y)r_f,$$

where y is the fraction invested in the risky portfolio P.

A **risk-neutral** investor only cares about maximizing expected return:

$$E[R_C] = r_f + y\big(E[R_P] - r_f\big).$$

Therefore:

- If $E[R_P] > r_f$, expected return increases with y, so the investor would choose the largest feasible y.
- If $E[R_P] = r_f$, the investor is indifferent over y.
- If $E[R_P] < r_f$, the investor would choose the smallest feasible y.

In our estimates, $E[R_P]$ exceeds r_f in A5/A7/A8, so a risk-neutral investor would allocate as much as possible to the risky portfolio P, subject to practical borrowing constraints.

A11. Compare Sharpe ratios in A4 and A5. Which is higher and why?

From our computations:

$$\text{Sharpe}_{ann}^{A4} \approx 0.6055, \qquad \text{Sharpe}_{ann}^{A5} \approx 0.9725.$$

Therefore, the portfolio in **A5** achieves the higher Sharpe ratio.

Why? The Sharpe ratio is

$$\text{Sharpe} = \frac{E[R_P] - r_f}{\sigma_P},$$

so it increases when the portfolio has a higher expected excess return and/or lower volatility. Relative to A4, A5 uses a *larger opportunity set* (MSFT, AGG, VNQ, and the market proxy SPY, plus the risk-free rate), whereas A4 is restricted to only two risky assets (MSFT and AGG) and also imposes weight bounds.

With more assets, the optimizer can exploit diversification and covariance structure more effectively (including hedging via negative weights when unconstrained), which can reduce portfolio risk for a given level of expected excess return (or increase expected excess return for a given level of risk). In mean–variance terms, expanding the set of available risky assets cannot lower the maximum achievable Sharpe ratio in theory; it can only weakly increase it. Consistent with this, our estimated tangency portfolio in A5 has a substantially higher Sharpe ratio than the constrained two-asset tangency portfolio in A4.

A12. Compare the results in A7 and A8. Which portfolio has higher Sharpe ratios? Why?

In A7 (daily data, Sep 1, 2023 to Aug 31, 2025), the tangency portfolio achieves an annualized Sharpe ratio of

$$\text{Sharpe}_{ann}^{A7} \approx 0.9839.$$

In A8 (weekly data over the same calendar window), the tangency portfolio achieves

$$\text{Sharpe}_{ann}^{A8} \approx 1.1036.$$

Thus, **A8 has the higher Sharpe ratio** in our sample.

Why might weekly data produce a higher Sharpe estimate? A key reason is that mean–variance (and tangency) portfolios are very sensitive to estimated inputs. Switching from daily to weekly returns changes both the estimated expected excess returns and the estimated covariance matrix. Weekly returns tend to be less affected by daily microstructure noise (bid–ask bounce), non-synchronous trading effects, and very short-horizon idiosyncratic fluctuations, which can lead to a different (and sometimes more stable) covariance estimate. In turn, the optimizer can find a portfolio with a higher estimated risk-adjusted performance.

In addition, using weekly data reduces the number of observations, which can change sampling variation in the estimated mean vector $\hat{\mu}$ and covariance matrix $\hat{\Sigma}$. Because tangency weights depend on $\hat{\Sigma}^{-1}(\hat{\mu} - \bar{r}_f \mathbf{1})$, even modest differences in estimated inputs can translate into noticeable differences in the Sharpe ratio. Therefore, while A8 has the higher Sharpe ratio in this sample, the difference should be interpreted cautiously as an *estimate* that depends on frequency and sampling.

B1. Life-cycle changes in the complete portfolio ("Portfolio C")

For an investor with a secure Ontario government job, the key life-cycle idea is that her *human capital* (the present value of future labor income) is relatively stable and bond-like. Early in her career, she has many years of labor income ahead, so human capital is large relative to financial wealth. Because this labor income is relatively safe, she can typically afford to take more financial risk: the complete portfolio C tends to have a higher weight on the risky portfolio (equities/REITs/market index) and a lower weight on the risk-free asset.

As the investor approaches retirement, the remaining labor income stream shrinks (human capital declines) and her financial wealth becomes a larger share of total wealth. She also has less time to recover from market downturns. Therefore, the complete portfolio C tends to become more conservative over time: the weight on the risky portfolio declines and the weight on the risk-free asset (T-bills) and/or safer fixed-income exposure increases. Intuitively, she shifts from growth-oriented investing early on toward capital preservation and stability later on.

B2. Is our broad-based market index a good proxy for the Market Portfolio?

We use SPY (S&P 500 ETF) as our broad-based market index. SPY is a reasonable practical proxy for the market portfolio because it is well diversified across large U.S. firms, highly liquid, and widely used as a benchmark. In empirical CAPM work, broad equity indexes like the S&P 500 are commonly used as an observable stand-in for the (unobservable) market portfolio.

However, SPY is not the true Market Portfolio in the theoretical CAPM sense. The Market Portfolio should include *all* risky assets held by investors, not just U.S. large-cap equities. In reality that includes international equities, bonds, real estate, private businesses, commodities, and possibly human capital. Also, because we are a Canada-based setting, a Canadian or global market index might better reflect the opportunity set of a typical Canadian investor. Hence SPY is a useful and standard proxy, but it is an imperfect approximation to the full Market Portfolio.

B3. Beta estimates (daily data, Sep 1 2023 to Aug 31 2024)

We estimate CAPM betas using the regression

$$(r_{i,t} - r_{f,t}) = \alpha_i + \beta_i \, (r_{m,t} - r_{f,t}) + \varepsilon_{i,t},$$

where $r_{i,t}$ is the daily continuously compounded (log) return on asset i, $r_{m,t}$ is the market proxy (SPY), and $r_{f,t}$ is the daily log risk-free rate (3-month T-bill). The slope coefficient β_i measures the asset's systematic risk relative to the market.

Table 9: B3: CAPM regressions using daily excess returns, Sep 1 2023 to Aug 31 2024.

Asset	$\hat{\alpha}$	$\hat{\beta}$	$p(\alpha)$	$p(\beta)$	R^2
MSFT	-1.9986×10^{-4}	1.1047	0.7326	1.10×10^{-34}	0.4646
AGG	-2.1485×10^{-5}	0.1455	0.9324	1.56×10^{-5}	0.0744
VNQ	4.7740×10^{-5}	0.8683	0.9381	1.39×10^{-22}	0.3270
SPY	1.1675×10^{-7}	0.99995	0.9112	0.0000	0.999995

The estimates are economically sensible: MSFT has $\beta > 1$ (more sensitive than the market), VNQ has a high equity-like beta ($\beta \approx 0.87$), while AGG has a low beta ($\beta \approx 0.15$), consistent with bond returns being much less correlated with equity market movements. The alpha estimates are statistically insignificant (high p-values), while betas for MSFT, AGG, and VNQ are statistically significant.

B4. Robustness of beta

We assess robustness by re-estimating CAPM betas under alternative sampling frequencies and checking whether the estimated β values are stable in sign and magnitude. We compare (i) daily data and (ii) weekly data over the same long window (Sep 1, 2022 to Aug 31, 2025). Betas are estimated from:

$$(r_{i,t} - r_{f,t}) = \alpha_i + \beta_i(r_{m,t} - r_{f,t}) + \varepsilon_{i,t},$$

where the market proxy is SPY and the risk-free rate is the (log) 3-month T-bill rate (daily log rate for daily returns; summed daily log rates within each week for weekly returns).

Table 10: B4: CAPM betas using **daily** excess returns, Sep 1 2022 to Aug 31 2025.

Asset	$\hat{\alpha}$	$\hat{\beta}$	$p(\alpha)$	$p(\beta)$	R^2
MSFT	1.3152×10^{-4}	1.1065	0.7481	1.35×10^{-123}	0.5363
AGG	-4.7355×10^{-5}	0.0842	0.7527	2.84×10^{-9}	0.0474
VNQ	-4.3331×10^{-4}	0.7793	0.2079	3.78×10^{-96}	0.4485

Table 11: B4: CAPM betas using **weekly** excess returns, Sep 1 2022 to Aug 31 2025.

Asset	$\hat{\alpha}$	$\hat{\beta}$	$p(\alpha)$	$p(\beta)$	R^2
MSFT	8.1716×10^{-4}	0.9781	0.6730	4.42×10^{-21}	0.4456
AGG	-5.4102×10^{-4}	0.0280	0.4301	9.94×10^{-3}	0.0432
VNQ	-2.2214×10^{-3}	0.8208	0.1555	2.53×10^{-22}	0.4660

Interpretation. The qualitative ranking of systematic risk is stable across the two specifications: MSFT has the highest beta, VNQ has a substantial equity-like beta, and AGG has a very low beta. Quantitatively, MSFT's beta decreases from about 1.11 (daily) to about 0.98 (weekly), while VNQ's beta increases modestly from about 0.78 (daily) to about 0.82 (weekly). AGG remains close to zero in both cases (0.084 daily vs 0.028 weekly). These differences are consistent with frequency effects and return aggregation: weekly data reduce high-frequency noise and can alter covariances, while the overall economic conclusions (bonds low beta; equity/REIT higher beta) remain robust.

B5. Choice of the risk-free rate

We represent the risk-free rate using the Bank of Canada 3-month Treasury bill yield (BoC series V39065), converted to a daily continuously compounded rate to match our log-return framework.

Is this rate truly risk-free? In practice, a Government of Canada T-bill is commonly treated as default-free in CAD because it is backed by the federal government and has negligible default probability over short horizons. However, it is not risk-free" in every sense: (i) it still has *inflation (real purchasing power) risk*; (ii) if an investor must liquidate before maturity, there can be *interest-rate / price risk*; (iii) there may be small *liquidity* and *reinvestment* risks; and (iv) the quoted yield is an annualized money-market yield that requires conversion/compounding assumptions to match the return frequency used in the analysis.

Does the investment horizon matter? Yes. The appropriate risk-free asset should match the investor's horizon as closely as possible. For a one-day (daily) horizon, the best theoretical risk-free asset is an overnight risk-free instrument, while for a one-month horizon a one-month T-bill is closer. Using a 3-month T-bill as risk-free" is a reasonable practical proxy for short-horizon analysis, but horizon mismatch can matter if term premia or rate volatility are non-negligible. In particular, a longer-maturity bill is more exposed to interest-rate changes if sold before maturity, so it is closer to risk-free only when held to maturity over a similar horizon.

B6. Beta of the T-bill

In the CAPM, beta is defined for *excess* returns. For the risk-free asset (T-bill), the excess return is

$$r_{f,t} - r_{f,t} = 0 \quad \text{for all } t.$$

Therefore, regressing the T-bill excess return on the market excess return yields a slope (beta) of zero:

$$(r_{f,t} - r_{f,t}) = \alpha_f + \beta_f (r_{m,t} - r_{f,t}) + \varepsilon_t.$$

Using data from Sep 1, 2024 to Aug 31, 2025, we obtain $\hat{\alpha} = 0$ and $\hat{\beta} = 0$ (up to numerical rounding). The reported R^2 is not meaningful here because the dependent variable is identically zero, so its variance is zero and R^2 becomes undefined in the usual formula.

B7. Expected returns: A5 optimal risky portfolio vs. T-bill (Sep 1, 2024–Aug 31, 2025)

We compute expected returns over Sep 1, 2024 to Aug 31, 2025 using daily continuously compounded (log) returns. For the A5 optimal risky portfolio, we apply the A5 weights to the four risky-asset daily log return series and take the sample mean. For the T-bill, we take the sample mean of the daily log risk-free return series. Annualization uses 252 trading days:

$$\mu_{\text{ann}} = 252\,\mu_d.$$

Table 12: B7: Expected returns (daily log mean and annualized mean) for Sep 1, 2024–Aug 31, 2025.

	Daily mean (μ_d)	Annualized mean (μ_{ann})
A5 optimal risky portfolio	0.0004812668	0.1212792
T-bill (risk-free)	0.0001194241	0.03009487

B8. Show that the slope of the CAL equals the Sharpe ratio

Consider a complete portfolio C formed by investing a fraction y in a risky portfolio P and $1-y$ in the risk-free asset with return R_f. The expected return of C is

$$\mathbb{E}[R_C] = (1-y)R_f + y\,\mathbb{E}[R_P] = R_f + y\big(\mathbb{E}[R_P] - R_f\big).$$

Because the risk-free asset has zero variance and zero covariance with risky returns, the standard deviation of C is

$$\sigma_C = y\,\sigma_P.$$

Therefore the slope of the capital allocation line (CAL), defined as excess expected return per unit risk, is

$$\frac{\mathbb{E}[R_C] - R_f}{\sigma_C} = \frac{y(\mathbb{E}[R_P] - R_f)}{y\sigma_P} = \frac{\mathbb{E}[R_P] - R_f}{\sigma_P}.$$

The right-hand side is the Sharpe ratio of portfolio P. Hence, the slope of the CAL equals the Sharpe ratio.

B9. Slope of the CAL for the optimal risky portfolio in A5

From B8, the slope of the capital allocation line (CAL) associated with a risky portfolio P is its Sharpe ratio:

$$\text{slope} = \frac{\mathbb{E}[R_P] - R_f}{\sigma_P}.$$

Using the A5 optimal risky portfolio moments (annualized with 252 trading days),

$$\mu_{P,\text{ann}} = 0.1254883, \qquad \sigma_{P,\text{ann}} = 0.09808966, \qquad r_{f,\text{ann}} = 0.03009157,$$

the slope of the CAL is

$$\text{slope} = \frac{\mu_{P,\text{ann}} - r_{f,\text{ann}}}{\sigma_{P,\text{ann}}} = \frac{0.1254883 - 0.03009157}{0.09808966} = 0.9725463.$$

Hence, the CAL for the A5 optimal risky portfolio has slope (Sharpe ratio) approximately 0.973.

B10. Current Stock Price and CAPM-Implied Expected Return

Current price (MSFT). Using `getQuote("MSFT")` in R, the last traded price is \$470.28 (timestamp: 2026–01–26 16:00:02).

CAPM market-consensus expected return. Using the CAPM,

$$\mathbb{E}[R_i] = r_f + \beta_i\big(\mathbb{E}[R_m] - r_f\big),$$

and our estimated inputs from the project (annualized),

$$r_f = 0.03009487, \quad \mathbb{E}[R_m] = 0.148992982, \quad \beta_{\text{MSFT}} = 1.1046974,$$

we obtain

$$\mathbb{E}[R_{\text{MSFT}}] = 0.03009487 + 1.1046974(0.148992982 - 0.03009487) = 0.1614413.$$

Thus, the CAPM-implied expected annual return for MSFT is approximately **16.14%**.

B11. CAPM Alpha (Jensen's Alpha)

We estimate the CAPM regression using excess returns:

$$(R_i - r_f) = \alpha_i + \beta_i(R_m - r_f) + \varepsilon_i.$$

Thus, the CAPM alpha is the intercept α_i. Using the B3 regression results for MSFT,

$$\hat{\alpha}_{\text{daily}} = -0.0001998644, \qquad \hat{\alpha}_{\text{annual}} = 252 \times \hat{\alpha}_{\text{daily}} = -0.05036583 \ (\approx -5.04\%).$$

The alpha is not statistically significant (p-value = 0.7326). The regression fit is $R^2 = 0.4646$.

References

1. Bank of Canada. Selected Treasury Bill Yields (V39065)" (daily series). Accessed: March 5, 2026.

2. Yahoo Finance. Historical price data for tickers used in this project (e.g., MSFT, AGG, VNQ, SPY). Accessed: March 5, 2026.

3. Course materials for ECON 3346 (portfolio/CAPM topics, definitions, and formulas). Western University.

Appendix

Zip folder with all codes and calculations attached